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Time-space non-commutativity in gravitational quantum well scenario

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Abstract. A novel approach to the analysis of the gravitational well problem from the point of view of a second quantized description is discussed. The second quantized formalism enables us to study the effect of time-space non-commutativity in the gravitational well scenario; this study is hitherto unavailable in the literature. The corresponding first quantized theory reveals a leading order perturbation term of non-commutative origin. The latest experimental findings are used to estimate an upper bound on the time-space non-commutative parameter. Our results are found to be consistent with the order of magnitude estimations of other NC parameters reported earlier.

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1 Introduction

The idea of non-commutative (NC) space-time where the coordinates x^μ satisfy the non-commutative algebra

$$[x^\mu, x^\nu] = \mathrm{i}\Theta^{\mu\nu} \tag{1}$$

has gained prominence in the recent literature. Originally mooted by Snyder in a different perspective [1, 2], this idea has been revived in the recent past [3, 4], and field theories defined over this NC space are currently subject to very intense research [5]. A wide range of theories are being formally studied in a NC perspective, encompassing various gauge theories [6–12], including gravity [13–21]. Apart from studying the formal aspects of the NC geometry, certain possible phenomenological consequences have also been investigated [22–35]. A part of the endeavor is spent in finding the order of the NC parameter and in exploring its connection with observations [36–38].

A particular piece of the scenario is the quantum well problem, which has emerged in recent GRANIT experiments by Nesvizhevsky et al. [39–41], who detected the quantum states of neutrons trapped in the earth's gravitational field. Their results have been used by Bertolami et al. [42, 43] and Banerjee et al. [44] to set an upper bound on the momentum space NC parameters. These works have been done on the level of quantum mechanics (QM), where non-commutativity is introduced by means of the phase space variables. Naturally, non-commutativity in the time-space sector cannot be accounted for in this picture, since in QM as such, space and time could not be treated on an equal footing. Time-space non-commutativity, however, has gained considerable interest in the current literature, and the search for any possible upper bound on the time-space NC parameter using recent experimental feedback is very desirable.

The issue of time-space non-commutativity is worth pursuing in its own right because of its deep connection with such fundamental notions as unitarity and causality. It was argued that the introduction of time-space non-commutativity spoils unitarity [45, 46] or even causality [47]. Much attention has been devoted in recent times to circumvent these difficulties in formulating theories with $\theta^{0\mathrm{i}} \neq 0$ [48–51]. In [52], it was shown in the context of the NC Schwinger model in $(1+1)$-dimensions that, in a perturbative approach, retaining terms up to first order in the NC parameter does give physically meaningful results. There are similar examples of other theories with time-space non-commutativity in the literature [53–55], where unitarity is preserved by an order by order perturbative approach.

In the present letter we shall study the effect of time-space NC (if any) on the energy spectrum of a cold neutron trapped in a gravitational quantum well by restricting ourselves to a first order perturbative treatment. To introduce time-space non-commutativity in this quantum well scenario, a second quantized theory is required. We propose to discuss the NC quantum well problem reducing it from a NC Schrödinger field theory. This is a reasonable starting point, since single particle quantum mechanics can be viewed as the one-particle sector of quantum field theory in the very weakly coupled limit; in this case the Schrödinger wave function essentially obeys the field equations [56–58]. This allows us to examine the effect of the whole sector of space-time non-commutativity in an

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effective non-commutative quantum mechanical (NCQM) theory. We do not consider momentum space NC effects as has been done by [42–44]. Our aim is to study the effect of non-commutativity on the level of quantum mechanics if time-space non-commutativity is accounted for.

The organization of the letter is the following. In the next section we consider a NC Schrödinger field interacting with an external classical gravitational field. We show that the canonical structure of the effective commutative theory can be revived with suitable mass and field rescaling. Both the Lagrangian and the Hamiltonian formulation is discussed. Once the canonical form is obtained, we get back to the first quantized level in Sect. 3. Here the ordinary quantum mechanics of the gravitational well problem is briefly reviewed, before we consider the effective NCQM and work out the perturbed energy spectrum in three different approaches. Here, much in the spirit of [42, 44], we use the experimental results of [39–41] to work out an estimation of the highest possible value of the time-space NC parameter. In Sect. 4, we make a rough calculation to show the consistency of our result with the estimations of other NC parameters existing in the literature [42, 44]. We draw our conclusions in Sect. 5.

2 The NC Schrödinger action

In this section we shall consider a NC field theory of a non-relativistic system with a constant background interaction. Evidently, the starting point is to write down the NC Schrödinger action. Now, there are two standard approaches to carry out the analysis of NC field theories. One can work in a certain Hilbert space that carries a representation of the basic NC algebra, and the fields are defined in this Hilbert space by the Weyl–Wigner correspondence [5]. We choose to take the alternative approach, in which we work in the deformed phase space with the ordinary product replaced by the star product [52, 57, 59, 60]. In this formalism, the fields are defined as functions of the phase space variables, and the redefined product of the two fields $\hat{\phi}(x)$ and $\hat{\psi}(x)$ is given by

$$\hat{\phi}(x)\star\hat{\psi}(x)=\left(\hat{\phi}\star\hat{\psi}\right)(x)=\mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\alpha\beta}\partial_\alpha\partial'_\beta}\hat{\phi}(x)\hat{\psi}(x')\Big|_{x'=x}. \tag{2}$$

In the star-product formalism the action for the NC Schrödinger field $\hat{\psi}$ coupled with the background gravitational field reads

$$\hat{S}=\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\hat{\psi}^\dagger\star\left[\mathrm{i}\hbar\partial_0+\frac{\hbar^2}{2m}\partial_i\partial_i-mg\hat{x}\right]\star\hat{\psi}\,. \tag{3}$$

The quantities described above act on a system in the vertical xy $(i=1,2)$ plane, where the external gravitational field is taken parallel to the x-direction. Under the $\star$ composition, the Moyal bracket for the coordinates is

$$[\hat{x}^\mu,\hat{x}^\nu]_\star=\mathrm{i}\Theta^{\mu\nu}=\begin{pmatrix}0 & -\eta & -\eta' \\ \eta & 0 & \theta \\ \eta' & -\theta & 0\end{pmatrix}, \tag{4}$$

where the μ,ν take the values $0,1,2$. Spatial non-commutativity is denoted by $\Theta^{12}=\theta$ and non-commutativity among time and the two spatial directions are denoted by the parameters $\Theta^{10}=\eta$ and $\Theta^{20}=\eta'$. With the hindsight that any possible deformation in the ordinary theory due to non-commutativity is expected to be of a small magnitude, we expand the star product and consider only the first order correction terms, which considerably simplifies the analysis.

2.1 First order equivalent commutative theory

Expanding the $\star$-product to first order in the NC parameters (4), we get

$$\hat{S}=\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\psi^\dagger\left[\mathrm{i}\hbar\left(1-\frac{1}{2\hbar}mg\eta\right)\partial_t+\frac{\hbar^2}{2m}\partial_i^2\right. \\ \left.-mgx-\frac{\mathrm{i}}{2}mg\theta\partial_y\right]\psi\,, \tag{5}$$

where everything is put in terms of commutative variables, and the NC effect is manifest by the presence of the θ and η terms. Clearly, the standard form of the kinetic term of the Schrödinger action is deformed, due to the time-space non-commutativity. To make the matter simpler, we rescale the field variable by

$$\psi\mapsto\tilde{\psi}=\sqrt{\left(1-\frac{\eta}{2\hbar}mg\right)}\;\psi\,, \tag{6}$$

which gives the conventionally normalized kinetic term. Such physically irrelevant rescalings have been done earlier [38, 57]. Therefore, it becomes clear that it is $\tilde{\psi}$, rather than ψ, that corresponds to the basic field variable in the action (5). It is therefore desirable to re-express it in terms of $\tilde{\psi}$ and to ensure that it is in the standard form in the first pair of terms. We have

$$\hat{S}=\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\tilde{\psi}^\dagger \\ \times\left[\mathrm{i}\hbar\partial_t+\frac{\hbar^2}{2\tilde{m}}\partial_i^2-\tilde{m}\left(1+\frac{\tilde{m}g\eta}{\hbar}\right)gx\;-\frac{\mathrm{i}}{2}\tilde{m}g\theta\partial_y\right]\tilde{\psi}\,. \tag{7}$$

Note that the mass term has also been rescaled,

$$\tilde{m}=\left(1-\frac{\eta}{2\hbar}mg\right)m\,, \tag{8}$$

and we can interpret $\tilde{m}$ as the observable mass. A similar charge rescaling of NC origin in the context of NC QED was shown in [38]. The last term in (7) can be absorbed in the term ∂_y^2 by rewriting

$$\partial_y=\left(\partial_y-\frac{\mathrm{i}\theta}{2\hbar^2}\tilde{m}^2g\right), \tag{9}$$

and the final effective NC Schrödinger action reads

$$\hat{S}=\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\tilde{\psi}^\dagger\left[\mathrm{i}\hbar\partial_t+\frac{\hbar^2}{2\tilde{m}}\left(\partial_x^2+\partial_y^2\right)\right. \\ \left.-\tilde{m}gx-\eta\left(\frac{\tilde{m}^2g^2}{\hbar}\right)x\right]\tilde{\psi}\,. \tag{10}$$

The Lagrange equation of motion for the fundamental field $\tilde{\psi}(x)$ is

$$\left[\mathrm{i}\hbar\partial_t + \frac{\hbar^2}{2\tilde{m}}\left(\partial_x^2+\partial_y^2\right) - \tilde{m}gx - \eta\left(\frac{\tilde{m}^2g^2}{\hbar}\right)x\right]\tilde{\psi} = 0\,. \quad (11)$$

Note that owing to the field and mass redefinition (6) and (8), everything but the last term takes the form of the standard Schrödinger field equation.

2.2 Hamiltonian analysis

We can also start with the commutative equivalent action (5) and derive the field equation in a Hamiltonian formalism. The advantage of the Hamiltonian analysis is twofold. It gives us a suitable platform to look for the proper canonical pair of fields, as well as a cross-check for the field equations.

The canonical momenta corresponding to the field variable ψ are

$$\Pi_\psi(x) = \mathrm{i}\hbar\left(1 - \frac{1}{2\hbar}mg\eta\right)\psi^\dagger(x)\,. \quad (12)$$

Note that in the argument we collectively refer to both spatial coordinates by x. Writing the usual Poisson bracket (PB) of the canonical pair

$$\left\{\psi(x), \Pi^\dagger(x')\right\} = \delta^2(x-y)\,, \quad (13)$$

and using the Faddeev–Jackiw technique [61] we get the basic bracket,

$$\left\{\psi(x), \psi^\dagger(x')\right\} = -\frac{\mathrm{i}}{\hbar}\left(1+\frac{1}{2\hbar}mg\eta\right)\delta^2(x-x')\,. \quad (14)$$

This is the non-standard form of the PB relation, and as such it indicates that ψ cannot represent the basic field variable. Following a physically trivial field rescaling, (6), we get the usual PB structure:

$$\left\{\tilde{\psi}(x), \tilde{\psi}^\dagger(x')\right\} = -\frac{\mathrm{i}}{\hbar}\delta^2(x-x')\,, \quad (15)$$

which justifies our earlier argument that instead of the original fields ψ one should choose the rescaled field variables $\tilde{\psi}$ and $\tilde{\psi}^\dagger$ as the canonical pair of fields.

The Hamiltonian density is worked out using (5) and (12), and we have

$$\begin{aligned}\mathcal{H} &= \Pi_\psi\dot{\psi} - \mathcal{L}\\ &= -\frac{\hbar^2}{2m}\psi^\dagger\partial_i^2\psi + mg\psi^\dagger x\psi + \frac{\mathrm{i}}{2}mg\theta\psi^\dagger\partial_y\psi\,, \end{aligned}\quad (16)$$

and, rewritten in terms of the rescaled fields $\tilde{\psi}, \tilde{\psi}^\dagger$ and the mass $\tilde{m}$, it becomes

$$\mathcal{H}(x) = -\frac{\hbar^2}{2\tilde{m}}\tilde{\psi}^\dagger\partial_i^2\tilde{\psi} + \tilde{m}g\left(1+\eta\frac{\tilde{m}g}{\hbar}\right)\tilde{\psi}^\dagger x\tilde{\psi}\,, \quad (17)$$

where the last term in (16) is absorbed in ∂_y^2 as usual (9). This Hamiltonian density generates the time evolution of the system, thus:

$$\begin{aligned}\dot{\tilde{\psi}} &= \left\{\tilde{\psi}(x), \mathcal{H}(x')\right\}\\ &= -\frac{\mathrm{i}}{\hbar}\left[-\frac{\hbar^2}{2\tilde{m}}\partial_i^2 + \tilde{m}g\left(1+\eta\frac{\tilde{m}g}{\hbar}\right)x\right]\tilde{\psi}\,, \end{aligned}\quad (18)$$

which is the same as our Lagrange field equation (11).

3 Reduction to first quantized theory

So far we have been dealing with the second quantized formalism, where $\tilde{\psi}$ was the basic field variable of the theory. The aim was to impose non-commutativity in the time-space sector and study how it affects the system. We found out that the only non-trivial change in the Schrödinger equation indeed originates from the space-time non-commutativity. Specifically, it shows up only in the direction of the external gravitational field $\mathbf{g} = -g\mathbf{e}_x$. This result is in conformity with [42–44], where it is shown that it is momentum non-commutativity, not space coordinate non-commutativity, that shows up in first order computations. In [42–44], along with spatial non-commutativity, momentum space non-commutativity has been included as well. However, the treatment of these references essentially leaves a gap in the analysis, which we fill in here. Since first and second quantized formalisms are equivalent as far as Galilean systems are concerned, in the sequel of this letter we carry out an equivalent NC quantum mechanical analysis in the first quantized formalism.

In the first quantized version of the theory the Schrödinger field equation (11) or (18) will be treated as the quantum mechanical equation of motion, and the earlier field variable $\tilde{\psi}$ will be interpreted as the wave function. This is a quick and simple but standard procedure to reduce the field theoretic setup to one-particle quantum mechanics as has been illustrated in [56] for a general external potential. We begin by checking that $\tilde{\psi}$ does have an interpretation as probability amplitude and satisfies the continuity equation

$$\partial_0 j_0 + \partial_i j_i = 0 \quad (i=1,2)\,, \quad (19)$$

with the usual expressions for the probability density j_0 and the probability current j_i in terms of $\tilde{\psi}$. From (18) we easily read off the Hamiltonian:

$$H = H_0 + H_1 = \frac{1}{2\tilde{m}}\left(p_x^2+p_y^2\right) + \tilde{m}gx + \eta\frac{\tilde{m}^2g^2}{\hbar}x\,. \quad (20)$$

Note that the NC effect in the ordinary part H_0 is hidden in the mass and field redefinitions (6) and (8). Such rescalings are only viable in a region of space-time where the variation of the external field is negligible. Since the results we have derived are to be compared with the outcome of a laboratory-based experiment, we can safely assume a constant external gravitational field throughout.

Before proceeding with the Hamiltonian (20), we should note that even if the variables in the commutative equivalent model are commuting it is not obvious that the usual Hamiltonian procedure could produce the dynamics with respect to non-commuting time. To first order in the NC parameter it has been shown in [50] that time-space non-commutativity emerges from a duality transformation, and Hamiltonian analyses are identical for the original theory (with non-commutativity in the spatial sector only) and its dual containing space-time non-commutativity. Such an approach has been shown to lead to a reasonable outcome in [52]. Following these references, we propose to carry out our analysis to first order in η and assume the applicability of the usual Hamiltonian dynamics for the commutative equivalent model.

Since we expect that the time-space NC parameter is rather small at the quantum mechanical level, the last term in (20) represents a perturbation H_1 to the usual gravitational quantum well scenario described by H_0. We now briefly review the ordinary quantum well problem, its solutions and the experimental results [39, 40] before any further discussion of the NC extension.

3.1 Ordinary gravitational quantum well

The first two terms in (20) form the commutative Hamiltonian H_0 of the gravitational well problem, which describes the quantum states of a particle with mass $\tilde{m}$ trapped in a linear potential well, in this case a gravitational well. The system's wave function can be separated into two parts, corresponding to each of the coordinates x and y. Since the particle is free to move in the y-direction, its energy spectrum is continuous along y and the corresponding wave function can be written as a collection of plane waves,

$$\tilde{\psi}(y) = \int_{-\infty}^{+\infty} g(k)\mathrm{e}^{\mathrm{i}ky}\,\mathrm{d}k\,, \tag{21}$$

where the function $g(k)$ determines the shape of the wave packet in phase space. The analytical solutions of the Schrödinger equation in the x-direction, i.e. the solutions to the eigenvalue equation $H_0\tilde{\psi}_n = E_n\tilde{\psi}_n$, are well known [62]. The eigenfunctions corresponding to x can be expressed in terms of the Airy function $\phi(z)$,

$$\psi_n(x) = A_n\phi(z)\,, \tag{22}$$

with the eigenvalues determined by the roots of the Airy function, α_n, with $n = 1, 2\ldots$,

$$E_n = -\left(\frac{\tilde{m}g^2\hbar^2}{2}\right)^{1/3}\alpha_n\,. \tag{23}$$

The dimensionless variable z is related to the height x by means of the following linear relation:

$$z = \left(\frac{2m^2g}{\hbar^2}\right)^{1/3}\left(x - \frac{E_n}{\tilde{m}g}\right). \tag{24}$$

The normalization factor for the nth eigenstate is given by

$$A_n = \left[\left(\frac{\hbar^2}{2m^2g}\right)^{\frac{1}{3}}\int_{\alpha_n}^{+\infty}\mathrm{d}z\phi^2(z)\right]^{-\frac{1}{2}}. \tag{25}$$

The wave function for a particle with energy E_n oscillates below the classically allowed height $x_n = \frac{E_n}{\tilde{m}g}$, and above x_n it decays exponentially. This was realized experimentally by Nesvizhevsky et al. [39–41] who observed the lowest quantum state of neutrons in the earth's gravitational field. The idea of the experiment was to let cold neutrons flow with a certain horizontal velocity ($6.5\,\mathrm{m\,s^{-1}}$) through a horizontal slit formed between a mirror below and an absorber above. The number of transmitted neutrons as a function of the absorber height is recorded, and the classical dependence is observed to change into a stepwise quantum mechanical dependence at a small absorber height. Their results and a comparison with the theoretical values are given below. The experimentally found value of the classical height for the first quantum state is

$$x_1^{\mathrm{exp}} = 12.2 \pm 1.8(\mathrm{syst.}) \pm 0.7(\mathrm{stat.})\,(\mu\mathrm{m})\,. \tag{26}$$

The corresponding theoretical value can be determined from (23) for $\alpha_1 = -2.338$, yielding

$$x_1 = 13.7\,\mu\mathrm{m}\,. \tag{27}$$

This value is contained in the error bars and allows for maximum absolute shift of the first energy level with respect to the predicted values:

$$\Delta E_1^{\mathrm{exp}} = 6.55\times10^{-32}\,\mathrm{J} = 0.41\,\mathrm{peV}\,. \tag{28}$$

The values of the constants taken in these calculations are as follows:

$$\begin{aligned}\hbar &= 10.59\times10^{-35}\,\mathrm{J\,s}\\ g &= 9.81\,\mathrm{m\,s^{-2}}\\ \tilde{m} &= 167.32\times10^{-29}\,\mathrm{kg}\,.\end{aligned} \tag{29}$$

3.2 Analysis of the perturbed energy spectrum

Going back to the effective NCQM theory, we now analyze the perturbed system (20). The perturbative potential is given by

$$H_1 = \eta\left(\frac{\tilde{m}^2g^2}{\hbar}\right)x\,. \tag{30}$$

Interestingly, the occurrence of this perturbation term is a direct manifestation of time-space non-commutativity. This enables us to work out an upper bound for the time-space non-commutative parameter. Following the prescription of [42], we can demand that the correction due to (20) in the energy spectrum should be smaller than or equal to the maximum energy shift allowed by the experiment [39–41]. We work out the theoretical value of the energy shift in three independent ways.

3.2.1 Numerical method

First, we take the numerical approach similar to [42] and calculate the leading order energy shift of the first quantum state. It is just the expectation value of the perturbation potential, given by

$$\Delta E_1 = \eta\frac{\tilde{m}^2g^2}{\hbar}\int_0^{+\infty} \mathrm{d}x\tilde{\psi}_1^*(x)x\tilde{\psi}_1(x)$$
$$= \eta\frac{\tilde{m}^2g^2}{\hbar}\left[\left(\frac{2\tilde{m}^2g}{\hbar^2}\right)^{-\frac{2}{3}}A_1^2I_1+\frac{E_1}{\tilde{m}g}\right], \qquad (31)$$

where the integral I_1 is defined by

$$I_1 \equiv \int_{\alpha_1}^{+\infty} \mathrm{d}z\phi(z)z\phi(z)\,. \qquad (32)$$

The values of the first unperturbed energy level E_1 are determined by (23) with $\alpha_1 = -2.338$:

$$E_1 = 2.259\times 10^{-31}\,(\mathrm{J}) = 1.407\,(\mathrm{peV})\,. \qquad (33)$$

The normalization factor A_1 is calculated by (25). The integrals in (25) and (32) were numerically determined for the first energy level:

$$A_1 = 588.109\,,\quad I_1 = -0.383213\,. \qquad (34)$$

The first order correction in the energy level is given by

$$\Delta E_1 = 2.316\times 10^{-23}\eta\,(\mathrm{J})\,, \qquad (35)$$

Comparing with the experimentally determined value of the energy level from (28), we found that the bound on the time-space NC parameter is

$$|\eta| \lesssim 2.83\times 10^{-9}\,\mathrm{m}^2\,. \qquad (36)$$

3.2.2 WKB method

Avoiding numerical methods, one can analyze the energy spectrum using a quasiclassical approximation. The potential term in the unperturbed Hamiltonian H_0 in (20) is linear, and simply using the WKB method suffices. The first energy level is given by the Bohr–Sommerfeld formula:

$$E_1 = \left(\frac{9m}{8}\left[\pi\hbar g\left(1-\frac{1}{4}\right)\right]^2\right)^{\frac{1}{3}} \qquad (37)$$
$$= \alpha_1 g^{\frac{2}{3}}\,,\quad n = 1,2,3\ldots\,, \qquad (38)$$

with $\alpha_1 = \left(\frac{9m}{8}[\pi\hbar(1-\frac{1}{4})]^2\right)^{\frac{1}{3}}$. This approximation gives nearly the exact value for the first energy level,

$$E_1 = 2.23\times 10^{-31}\,(\mathrm{J}) = 1.392\,(\mathrm{peV})\,, \qquad (39)$$

to be compared to (33). Since the perturbation term H_1 in (20) is also linear in x, we can combine it with the potential term and rewrite the potential term as follows:

$$V(x) = \tilde{m}g'x = \tilde{m}g\left(1-\frac{\eta\tilde{m}}{\hbar}\right)x\,. \qquad (40)$$

Now using the modified acceleration g' from (40) in (38), the approximate shift in the energy value is obtained by a first order expansion in η:

$$E_1+\Delta E_1 = \alpha_1 g'^{\frac{2}{3}} = \alpha_1 g^{\frac{2}{3}}\left(1-\frac{\eta\tilde{m}g}{\hbar}\right)^{\frac{2}{3}}$$
$$= \alpha_1 g^{\frac{2}{3}}\left(1-\frac{2\eta\tilde{m}g}{3\hbar}\right)$$
$$= E_1 - \eta\left(\frac{2E_1\tilde{m}g}{3\hbar}\right)\,. \qquad (41)$$

Note that in [44] a similar modification of the gravitational acceleration has been made to accommodate the perturbation term in the potential. Using the values of $\tilde{m}, g, \hbar$ and E_1 from (29) and (39), we calculate the energy shift ΔE_1:

$$\Delta E_1 = 2.304\times 10^{-23}\eta\,(\mathrm{J})\,. \qquad (42)$$

Again, this is comparable with (35). So we get nearly the same upper bound on the time-space NC parameter as in (36) by comparison with the experimental value (28):

$$|\eta| \lesssim 2.843\times 10^{-9}\,\mathrm{m}^2\,. \qquad (43)$$

3.2.3 Virial theorem method

Another simple analytical approach to calculate the energy shift ΔE_1 is to use the virial theorem [63], which implies $\langle T\rangle = \frac{1}{2}\langle V\rangle$, where T and V are the kinetic and potential energies, respectively. Hence the total energy is given by $E = \frac{3}{2}\langle V\rangle$. The gravitational potential is $V = \tilde{m}gx$, which gives

$$\langle x\rangle = \frac{2E}{3\tilde{m}g}\,. \qquad (44)$$

Now the perturbation term is

$$H_1 = \eta\left(\frac{\tilde{m}^2g^2}{\hbar}\right)\langle x\rangle\,. \qquad (45)$$

Here using (44) we find the energy shift in the first energy level as

$$\Delta E_1 = -\eta\left(\frac{2E_1\tilde{m}g}{3\hbar}\right)\,, \qquad (46)$$

which reproduces the same expression for ΔE_1 as derived in (41). Hence, the upper bound on η, using the virial theorem method, is exactly the same as in (43).

This concludes our analysis of the perturbed energy spectrum of the gravitational quantum well problem. This analysis leads to the evaluation of an upper bound on the time-space NC parameter in three independent methods.

4 Comparison with existing results

Now that we have an order of magnitude estimation for the time-space NC parameter it is instructive to enquire whether it is in conformity with the estimates of the other

NC parameters, reported earlier [42–44]. In [42] the upper bound on the fundamental momentum scale was calculated to be

$$\Delta p \lesssim 4.82 \times 10^{-31}\,\mathrm{kg\,m\,s^{-1}}\,. \tag{47}$$

Since $E \approx \frac{p_y^2}{2\tilde{m}}$, we have

$$\Delta E \approx \frac{p_y}{\tilde{m}} \Delta p_y = v_y \Delta p_y \lesssim 31.33 \times 10^{-31}\,\mathrm{kg\,m^2\,s^{-2}}\,. \tag{48}$$

Here, we have used the value of $v_y = 6.5\,\mathrm{m\,s^{-1}}$ used by the GRANIT experiment group. Using this value of ΔE in the time–energy uncertainty relation $\Delta E \Delta t \geq \hbar$, we find

$$\Delta t \geq \frac{\hbar}{\Delta E} = 3.38 \times 10^{-4}\,\mathrm{s}\,. \tag{49}$$

Hence, the uncertainty in the time-space sector can be calculated using the results of [42] as

$$\Delta x \Delta t \sim 3.38 \times 10^{-18}\,\mathrm{m\,s}\,, \tag{50}$$

where, following [42], we have taken $\Delta x \simeq 10^{-15}$ m. On the other hand, in the present letter we have derived the upper bound on the parameter η:

$$\eta = -\mathrm{i}[x^1, x^0] \lesssim 2.843 \times 10^{-9}\,\mathrm{m}^2\,. \tag{51}$$

Restoring the c-factor in (51), we write the commutator in terms of the variables x and t:

$$-\mathrm{i}[x, t] = \frac{\eta}{c} = \lesssim 9.51 \times 10^{-18}\,\mathrm{m\,s}\,. \tag{52}$$

Using the generalized uncertainty theorem [64] for the commutation relation in (52), we can write

$$\Delta x \Delta t \geq \frac{1}{2}\frac{\eta}{c} \sim 4.75 \times 10^{-18}\,\mathrm{m\,s}\,. \tag{53}$$

Interestingly, the value of the upper bound on the time-space NC parameter as derived here turned out to be consistent with the results of [42–44]. However, one should keep in mind that this value is only meant in the sense of an upper bound and not the value of the parameter itself.

5 Conclusions

In this letter we have obtained an effective NCQM description for the gravitational well problem starting from a NC Schrödinger action coupled to an external gravitational field. The effective commutative field theory is shown to take the usual form, once the proper canonical pair of field variables are identified by a Hamiltonian analysis. The effect of the non-commutativity on the mass parameter appears naturally in the process. We reinterpret this one-particle field theory as a first quantized theory and obtain an effective NCQM description for a particle trapped in the earth's gravitational field. Interestingly, we observe that the external gravitational field has to be static and uniform in order to get a canonical form for the Schrödinger equation up to η-corrected terms, so that a natural probabilistic interpretation emerges.

The main object of our analysis is to study the gravitational quantum well problem, reducing it from a field theoretic setting, so that time-space non-commutativity may be included in a natural way. The singularly important outcome of our calculation is that it is the underlying time-space sector of the NC algebra that is instrumental in introducing non-trivial NC effects in the energy spectrum of the system to first order perturbative level. Following [42, 43], we demand that the calculated perturbation in the energy level should be less than or equal to the maximum energy shift allowed by the GRANIT experiment performed at Grenoble [39–41]. This comparison leads to an upper bound on the time-space NC parameter. This upper bound is shown to be consistent with the existing upper bound for the fundamental momentum scale in the literature.

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